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School of Engineering

FEM and Elasticity Theory

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EG3111 – Finite Element Analysis and Design

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2b. A FEM for elasticity based on energy

- As for any PDE, the equations of elasticity can be solved using Weighted Residual Methods, as introduced in section 1b.
- However, the equations of elasticity can also be solved using a physical model, and this approach is always used to formulate the elastic FEM.
- In this physical method, an expression for the total energy of the system, Π, is determined.
- The displacement field is selected such that it minimises the total energy, *e.g.,* if the DOF is c then the solution satisfies

$$\frac{\partial \Pi}{\partial c} = 0$$

2b. A FEM for elasticity based on energy

The energy of an elastically deformed body consists of two parts such that

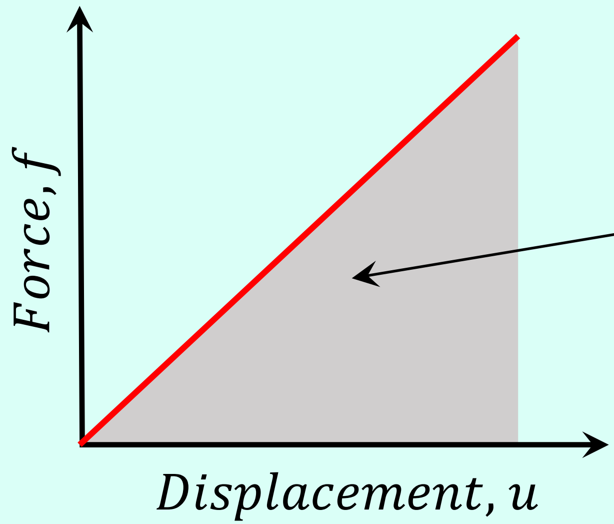
$$\Pi = U + \Omega$$

Where:

U is known as the elastic stored energy or the elastic strain energy

Ω is the potential energy of the applied loads

2b. Elastic strain energy, U



Work done, $U = Force \times Distance$

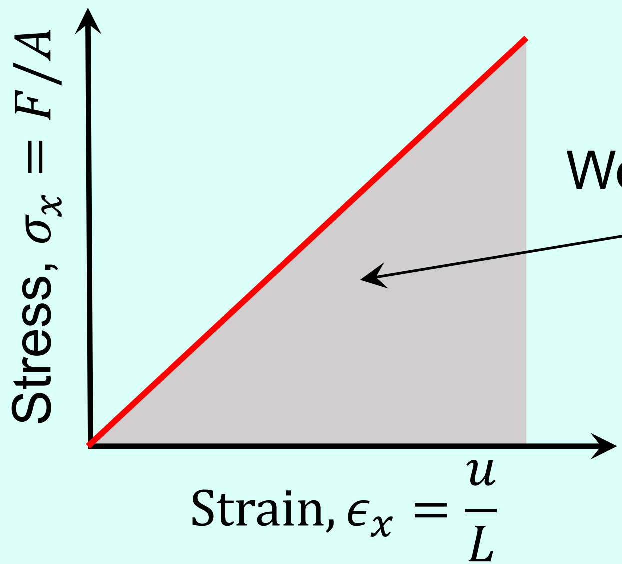
$= \text{area under the curve}$

$= \frac{1}{2} f u$

The factor of a half appears as the force increases with the displacement.

2b. Elastic strain energy density, w

Torsion spring



Work done per unit volume,

$$w = \frac{Force}{Area} \times \frac{Displacement}{Length}$$

$= \text{stress} \times \text{strain}$

$= \text{area under the curve}$

$= \frac{1}{2}\sigma_x \epsilon_x$

Total strain energy:

$$U = \int_V w\, dV = \frac{1}{2}\int_V \sigma_x \epsilon_x dV$$

Volume of the body

Typically the stress and strain (and hence w) vary throughout the body

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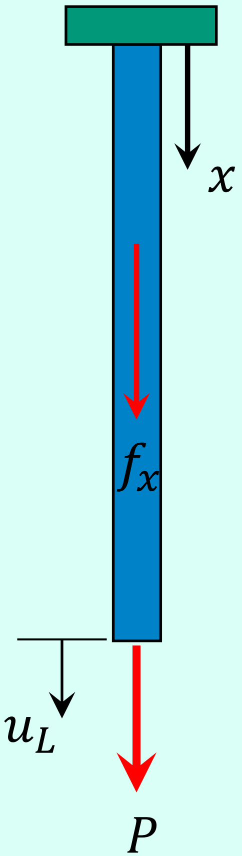
2b. Potential energy of the applied loads



Two types of contributions.

I. The **potential energy of the load *P*** (traction applied to the external end surface) is given by

$$\boldsymbol{\Omega = -P u_L}$$

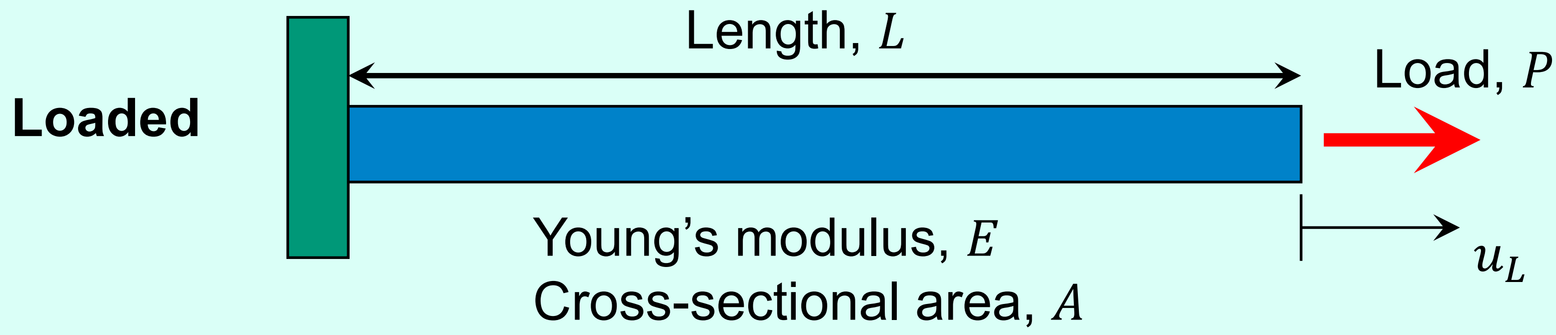
Where u_L is the distance moved by the applied load P.
(The minus sign shows that work is done by the load)

ii. The **potential energy of the body force** (force per unit volume)

$$\boldsymbol{\Omega = -\int_V f_x u \, dV}$$

(You can show that this is the change in total potential energy mgh of the body before and after loading).

2b. (i) Simple 1D extension using FEM



$$\Pi = U + \Omega$$

Where

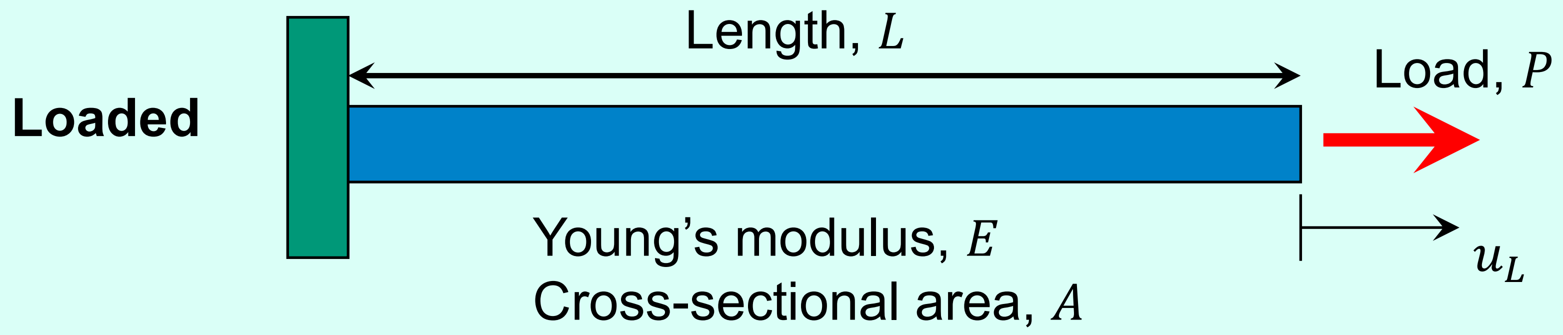
$$U = \int_V w \, dV = \frac{1}{2}\int_V \sigma_x \epsilon_x dV$$

$$\Omega = -Pu_L$$

Or

$$\Omega = -\int_V f_x u \, dV$$

2b. (i) Simple 1D extension using FEM



Propose a **linear shape function** for the solution

$$u(x) = u_L\left(\frac{x}{L}\right)$$

$$u(0) = 0$$

$$u(L) = u_L$$ Our DOF to be determined

One DOF is end displacement u_L

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2b. (i) Simple 1D extension using FEM

Elastic strain energy

$$U = \frac{1}{2}\int_V \sigma_x \epsilon_x dV$$

$$\epsilon_x = \frac{du}{dx} = \frac{u_L}{L} \qquad\qquad V = AL$$

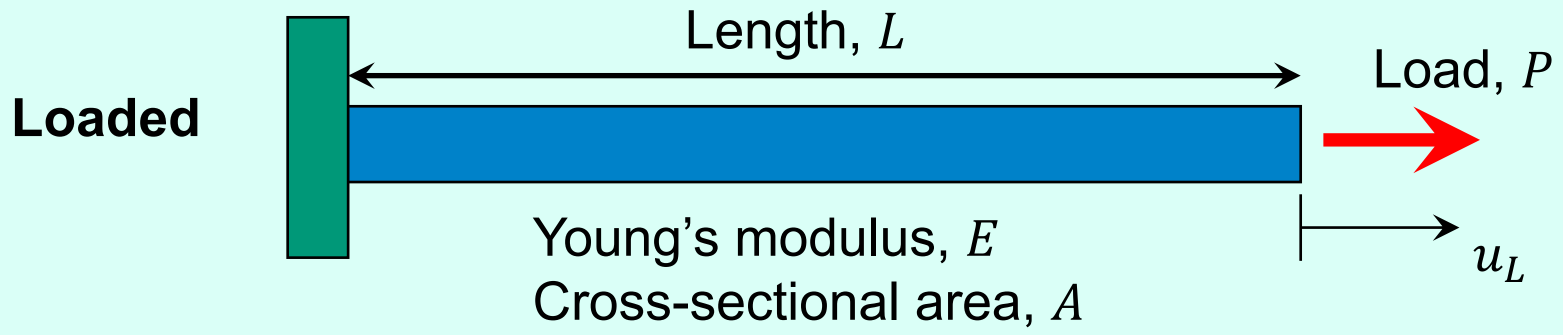
$$U = \frac{1}{2}\int_V \sigma_x \epsilon_x \, dV = \frac{1}{2} A \int_0^L E \epsilon_x^2 \, dx = \frac{1}{2} EA \int_0^L \epsilon_x^2 \, dx$$

$$U = \frac{1}{2} EA \int_0^L \left(\frac{u_L}{L}\right)^2 dx$$

$$U = \frac{1}{2}\frac{EAu_L^2}{L^2} L = \frac{1}{2}\frac{EA}{L} u_L^2$$

The elastic strain energy is always a second order polynomial in terms of DOF

2b. (i) Simple 1D extension using FEM



Potential energy of the applied load

$$\Omega = -Pu_L$$

The potential energy is always a first order polynomial in terms of DOF

2b. (i) Simple 1D extension using FEM

Total energy

$$\Pi = \frac{1}{2}\frac{EA}{L}u_L^2 - Pu_L \qquad Eq\ 4$$

Minimize total energy with respect to DOF

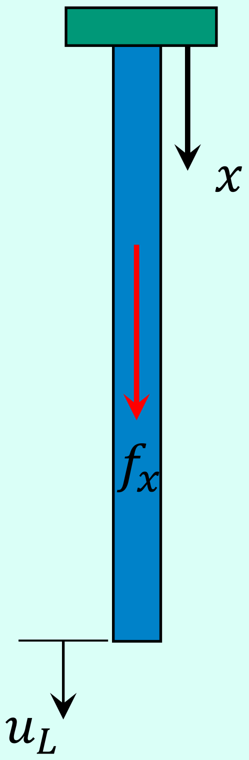
Differentiate $Eq\ 4$

$$\frac{\partial \Pi}{\partial u_L} = 0 \quad \Longrightarrow \quad \frac{EA}{L}\mathrm{u_L} - \mathrm{P} = 0$$

Same result as before….
Because the proposed shape function has the same form as the exact solution.

$$\Longrightarrow \mathrm{u_L} = \frac{\mathrm{PL}}{\mathrm{EA}} \quad \Longrightarrow \quad u(x) = \frac{\mathrm{P}}{\mathrm{EA}}\mathrm{x}$$

2b. (ii) Self-weight using FEM



Try solving this problem using:

I. Linear (1 DOF: u_1) shape function

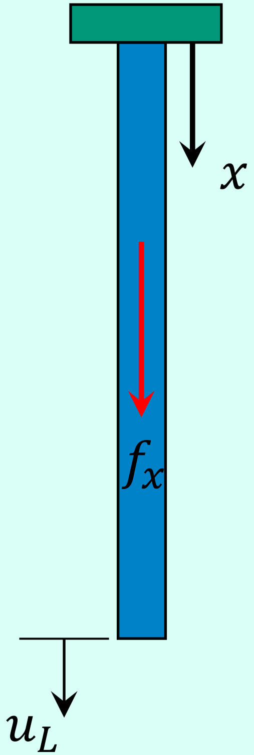
$$u(x) = u_1 \left(\frac{x}{L}\right)$$

ii. Quadratic (2 DOF: a and b) shape function

$$u(x) = a \left(\frac{x}{L}\right) + b \left(\frac{x}{L}\right)^2$$

2b. (ii) Self-weight (Linear Shape Function)

$$u(x) = u_1\left(\frac{x}{L}\right)$$



$$f_x = \rho g$$

$$U = \frac{1}{2}EA\int_0^L \epsilon_x^2\,dx$$

Propose

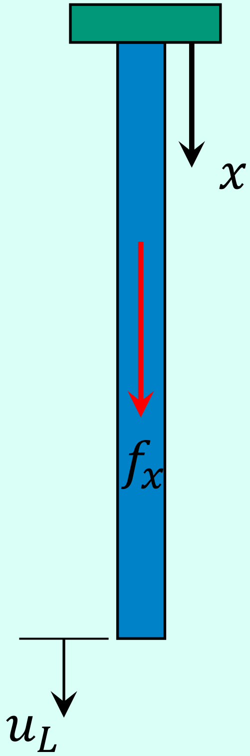
$$u(x) = u_1\left(\frac{x}{L}\right)$$

$$U = \frac{1}{2}EA\int_0^L \left(\frac{u_1}{L}\right)^2 dx = \frac{1}{2}\frac{EA}{L}u_1^2$$

$$\Omega = -\int_V f_x u(x)dV = -\int_0^L \rho g.u_1\left(\frac{x}{L}\right) Adx = -\frac{\rho g.u_1.A}{L}\int_0^L x\,dx$$

$$= -\frac{1}{2}A\rho gL.u_1$$

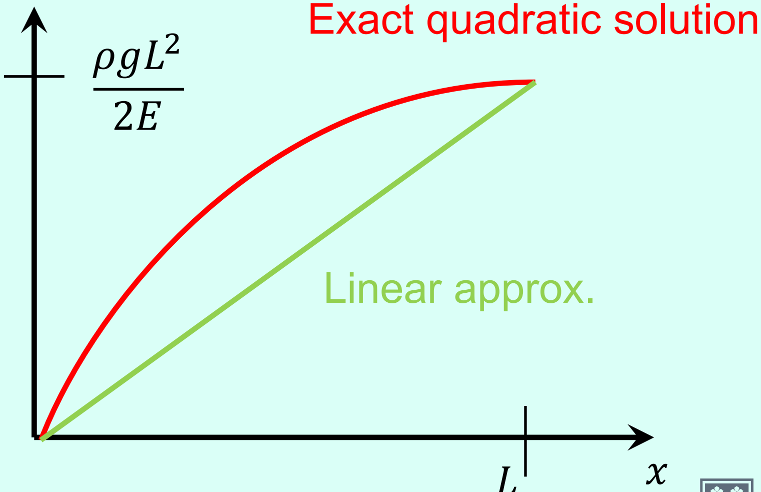
2b. (ii) Self-weight (Linear Shape Function)



$$\Pi = \frac{1}{2}\frac{EA}{L}u_1^2 - \frac{1}{2}A\rho gL.u_1$$

$$\frac{\partial \Pi}{\partial u_1} = 0 = \frac{EA}{L}u_1 - \frac{1}{2}A\rho gL = 0 \quad \Longrightarrow \quad u_1 = \frac{\rho g L^2}{2E}$$

$$u(x) = \frac{\rho g L}{2E}x$$



Next section…
(3) Bar Elements